A Smith Factorization Approach to Robust Minimum Variance Control of Nonsquare LTI MIMO Systems

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1. Introduction

The minimum phase systems have ultimately been redefined for LTI discrete-time systems, at first SISO and later square MIMO ones, as those systems for which minimum variance control (MVC) is asymptotically stable, or in other words those systems who have ’stable’ zeros, or at last as ‘stably invertible’ systems. The redefinition has soon been extended to nonsquare discrete-time LTI MIMO systems (Latawiec, 1998; Latawiec et al., 2000) and finally to nonsquare continuous-time systems (Hunek, 2003; Latawiec & Hunek, 2002), giving rise to defining of new ‘multivariable’ zeros, i.e. the so-called control zeros. Control zeros are an intriguing extension of transmission zeros for nonsquare LTI MIMO systems. Like for SISO and square MIMO systems, control zeros are related to the stabilizing potential of MVC, and, in the input-output modeling framework considered, are generated by (poles of) a generalized inverse of the ‘numerator’ polynomial matrix $B(.)$. Originally, the unique, so-called $T$-inverse, being the minimum-norm right or least-squares left inverse involving the regular (rather than conjugated) transpose of the polynomial matrix, was employed in the specific case of full normal rank systems (Hunek, 2002; Latawiec, 2004). The associated control zeros were later called by the authors ‘control zeros type 1’ (Latawiec, 2004; Latawiec et al., 2004), as opposed to an infinite number of ‘control zeros type 2’ generated by the myriad of possible polynomial matrix inverses, even those called $\tau$ - and $\sigma$ -inverses also involving the unique minimum-norm or least-squares inverses (Latawiec, 2004; Latawiec et al., 2005b). Transmission zeros, if any, are included in the set of control zeros; still, we will discriminate between control zeros and transmission zeros. In the later, new ‘inverse-free’ MVC design approach based on the extreme points and extreme directions method (Hunek, 2007; Hunek & Latawiec, 2006), it was possible to design a pole-free inverse of the polynomial matrix $B(.)$ so that no control zeros did appear. Well, except when transmission zeros are present, in which case the extreme points and extreme directions method does not hold. In the current important result of the authors, the Smith factorization of the polynomial matrix $B(.)$ can lead to its pole-free inverse, when there are no transmission zeros. Well, provided that the applied inverse is just the $T$-inverse, the
intriguing result bringing us back to the origin of the introduction of control zeros. And in case of any other inverse of Smith-factorized \( B() \) we end up with control zeros.

The remainder of this paper is organized as follows. System representations are reviewed in Section 2. Section 3 presents the problem of minimum variance control for discrete-time LTI MIMO systems. Section 4 describes the new approach to MVC design, confirming the Davison’s theory of minimum phase systems and indicates the role of the control zeros in robust MVC-related designs. A simple simulation example of Section 5 indicates favorable properties of the new method in terms of its contribution to robust MVC design. New results of the paper are summarized in the conclusions of Section 6.

2. System representations

Consider an \( n_u \)-input \( n_y \)-output LTI discrete- or continuous-time system with the input \( u(t) \) and the output \( y(t) \), described by possibly rectangular transfer-function matrix

\[
G \in \mathbb{R}^{n_y \times n_u}(p)
\]

in the complex operator \( p \), where \( p = z \) or \( p = s \) for discrete-time or continuous-time systems, respectively. The transfer function matrix can be represented in the matrix fraction description (MFD) form

\[
G(p) = A^{-1}(p)B(p),
\]

where the left coprime polynomial matrices \( A \in \mathbb{R}^{n_y \times n_y}[p] \) and \( B \in \mathbb{R}^{n_y \times n_u}[p] \) can be given in form

\[
A(p) = p^n I + \ldots + a_n p^{-n} \quad \text{and} \quad B(p) = p^m b_0 + \ldots + b_m p^{-m},
\]

respectively, where \( n \) and \( m \) are the orders of the respective matrix polynomials. An alternative MFD form

\[
G(p) = \tilde{B}(p)\tilde{A}^{-1}(p),
\]

involving right coprime \( \tilde{A} \in \mathbb{R}^{n_y \times n_y}[p] \) and \( \tilde{B} \in \mathbb{R}^{n_u \times n_y}[p] \), is also tractable here but in a less convenient way (Latawiec, 1998). Algorithms for calculation of the MFDs are known (Rosenbrock, 1970; Wolowich, 1974) and software packages in the MATLAB’s Polynomial Toolbox® are available. Unless necessary, we will not discriminate between

\[
A(p^{-1}) = I + \ldots + a_n p^{-n} \quad \text{and} \quad A(p) = p^n A(p^{-1}),
\]

nor between

\[
B(p^{-1}) = b_0 + \ldots + b_m p^{-m} \quad \text{and} \quad B(p) = p^m B(p^{-1}).
\]

In the sequel, we will assume for clarity that \( B(p) \) is of full normal rank; a more general case of \( B(p) \) being of non-full normal rank can be easily tractable (Latawiec, 1998). Let us finally concentrate on the case when normal rank of \( B(p) \) is \( n_u \) (‘symmetrical’ considerations can be made for normal rank \( n_u \)). Now, for discrete-time systems we have

\[
A(z^{-1}) = z^{-n} A(z) = I + \ldots + a_n z^{-n} \quad \text{and} \quad B(z^{-1}) = z^{-m} B(z) = b_0 + \ldots + b_m z^{-m},
\]

with \( G(z) = A^{-1}(z)B(z) = z^{-d} A^{-1}(z^{-1})B(z^{-1}) \), where \( d = n - m \) is the time delay of the system. The analyzed MFD form can be directly obtained from the AR(I)X/AR(I)MAX model of a system

\[
\bar{A}(q^{-1})y(t) = q^{-d} B(q^{-1})u(t) + [C(q^{-1})D(q^{-1})]v(t),
\]

where \( q^{-1} \) is the backward shift operator, \( y(t) \in \mathbb{R}^{n_y} \), \( u(t) \in \mathbb{R}^{n_u} \) and \( v(t) \in \mathbb{R}^{n_y} \) are the output, input and uncorrelated zero-mean disturbance, respectively, in (discrete) time \( t \); \( A \) and \( B \) as well as \( A \) and \( C \in \mathbb{R}^{n_y \times n_y}[z] \) are relatively prime polynomial matrices, with \( C(z^{-1}) = \xi_0 + \ldots + \xi_k z^{-k} \) and \( k \leq n \), and the \( D \) polynomial in

\[
\mathbb{R}^{n_y\times n_y}[z].
\]
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z^{-1} -domain is often equal to 1 - z^{-1} (or to n_y -identity matrix in discrete-time MVC considerations).

The familiar Smith-McMillan form \( S_M(p) \) (Kaczorek, 1998) of \( G(p) \) (as a special case of the MFD factorization (Desoer & Schulman, 1974)) is given by \( G(p) = U_0(p)S_M(p)V_0(p) \), where \( U_0 \in \mathbb{R}^{n_y \times y_n}[p] \) and \( V_0 \in \mathbb{R}^{n_u \times u_n}[p] \) are unimodular and the pencil \( S_M \in \mathbb{R}^{n_y \times n_x}(p) \) is of the form

\[
S_M(p) = \begin{bmatrix}
M_{r \times r} & 0_{r \times (n_u - r)} \\
0_{(n_y - r) \times r} & 0_{(n_y - r) \times (n_u - r)}
\end{bmatrix}
\]

with \( M(p) = \text{diag}(\epsilon_1 / \psi_1, \epsilon_2 / \psi_2, ..., \epsilon_r / \psi_r) \), where \( \epsilon_i(p) \) and \( \psi_i(p) \), \( i = 1, ..., r \) (with \( r \) being the normal rank of \( G(p) \)), are monic coprime polynomials such that \( \epsilon_i(p) \) divides \( \epsilon_{i+1}(p) \), \( i = 1, ..., r - 1 \), and \( \psi_i(p) \) divides \( \psi_{i-1}(p) \), \( i = 2, ..., r \). The particular Smith form is given by the appropriate pencil \( S(p) \), with \( M(p) = \text{diag}(\epsilon_1, \epsilon_2, ..., \epsilon_r) \) often associated with Smith zeros or transmission zeros. The polynomials \( \epsilon_i(p) \) are often called the invariant factors of \( G(p) \) and their product \( \epsilon(p) = \prod \epsilon_i(p) \) is sometimes referred to as the zero polynomial of \( G(p) \).

Extend the discrete-time system input-output description to the form accounting for additive disturbances

\[
A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + C(q^{-1})v(t)
\]

where \( u(t) \) and \( y(t) \) are the input and output vectors, respectively, \( v(t) \) is the zero-mean uncorrelated disturbance vector, \( d \) is the time delay and \( A(q^{-1}) \), \( B(q^{-1}) \) and \( C(q^{-1}) \) are the appropriate matrix polynomials (in the backward shift operator \( q^{-1} \)) of orders \( n \), \( m \) and \( k \), respectively. As usual, we assume that the leading coefficient of \( A(q^{-1}) \) is equal to the identity matrix. Assume that \( A(q^{-1}) \) and \( B(q^{-1}) \) as well as \( A(q^{-1}) \) and \( C(q^{-1}) \) are left coprime, with \( B(q^{-1}) \) and (stable) \( C(q^{-1}) \) being of full normal rank \( n_y \). For the general purposes and for duality with the continuous-time case, we use here the ARMAX model, even though it is well known that the \( C(q^{-1}) \) polynomial matrix of disturbance parameters is usually in control engineering practice unlikely to be effectively estimated (and is often used as a control design, observer polynomial matrix instead).

In the sequel, we proceed with discrete-time systems only but all the results are available for continuous-time systems as well (Hunek, 2003; Hunek & Latawiec, under review; Latawiec, 2004; Latawiec & Hunek, 2002; Latawiec et al., 2004).

3. Closed-loop minimum variance control

Consider a right-invertible system described by equation (2) and assume that the observer (or disturbance-related) polynomial \( C(q^{-1}) = c_0 + c_1 q^{-1} + ... + c_k q^{-k} \) has all roots inside the unit disk. (Note: Similar results can be obtained for left-invertible systems.)
Then the general MVC law, minimizing the performance index

$$\min_{u(t)} E \left[ \left( y(t+d) - y_{ref}(t+d) \right)^T \left( y(t+d) - y_{ref}(t+d) \right) \right]$$

(3)

where \( y_{ref}(t+d) \) and \( y(t+d) = \tilde{C}(q^{-1}) y_{ref}(t+d) - \tilde{H}(q^{-1}) y(t) + \tilde{E}(q^{-1}) \psi(t) \) (Hunek, 2003; Hunek, 2007; Hunek & Latawiec, 2006; Hunek & Latawiec, under review; Latawiec, 2004) are the output reference/setpoint and the stochastic output predictor, respectively, is of form

$$u(t) = B^R(q^{-1})^{-1} (q^{-1}) \tilde{C}(q^{-1}) y_{ref}(t+d) - \tilde{H}(q^{-1}) y(t)$$

(4)

The appropriate \( n_y \times n_y \)-polynomial matrices \( \tilde{F}(q^{-1}) = I_{n_y} + \tilde{f}_1 q^{-1} + ... + \tilde{f}_{d-1} q^{-d+1} \) and \( \tilde{H}(q^{-1}) = \tilde{h}_0 + \tilde{h}_1 q^{-1} + ... + \tilde{h}_{n-1} q^{-n+1} \) are computed from the polynomial matrix identity (called Diophantine equation)

$$\tilde{C}(q^{-1}) = \tilde{F}(q^{-1}) A(q^{-1}) + q^{-d} \tilde{H}(q^{-1})$$

(5)

with

$$\tilde{C}(q^{-1}) = \tilde{F}(q^{-1}) A(q^{-1})$$

(6)

where \( F(q^{-1}) = I_{n_y} + f_1 q^{-1} + ... + f_{d-1} q^{-d+1} \), \( \tilde{C}(q^{-1}) = \tilde{c}_0 + \tilde{c}_1 q^{-1} + ... + \tilde{c}_k q^{-k} \) and \( I_{n_y} \) denotes the \( n_y \)-identity matrix.

For right-invertible systems the symbol \( B^R(q^{-1}) \) denotes three possible classes of minimum-norm right \( T \), \( \tau \) - and \( \sigma \)-inverses of the polynomial matrix \( B(q^{-1}) \) (Hunek, 2003; Latawiec, 2004; Latawiec et al., 2004; Latawiec et al., 2005b; Latawiec et al., 2003). Like with transmission zeros for SISO and square MIMO systems, poles of the right inverse \( B^R(q^{-1}) \) have been defined as control zeros (Hunek, 2003; Hunek, 2007; Hunek & Latawiec, 2006; Hunek & Latawiec, under review; Latawiec, 1998; Latawiec, 2004; Latawiec et al., 2000; Latawiec & Hunek, 2002; Latawiec et al. 2005a; Latawiec et al., 2004; Latawiec et al., 2005b; Latawiec et al., 2003). The minimum-norm right inverse was used in the unique \( T \)-inverse to generate a unique set of control zeros type 1 for right-invertible systems (Hunek, 2003; Latawiec, 2004; Latawiec & Hunek, 2002; Latawiec et al., 2003).

However, the formula (4) can be treated as a solver of the MVC-related matrix polynomial equation

$$B(q^{-1}) u(t) = \tilde{y}(t)$$

(7)

where \( \tilde{y}(t) = \tilde{E}(q^{-1}) \tilde{C}(q^{-1}) y_{ref}(t+d) - \tilde{H}(q^{-1}) y(t) \) (Latawiec, 2004). When analyzing possible solutions to equation (7) we have introduced new classes of inverses of polynomial
matrices, that is a finite number of \( \tau \)-inverses and an infinite number of \( \sigma \)-inverses (Latawiec, 2004; Latawiec et al., 2004; Latawiec et al., 2005b), all surprisingly employing the unique minimum-norm right inverse. The \( \tau \)- and \( \sigma \)-inverses contribute to generation of what has been referred to as control zeros type 2 (Latawiec, 2004; Latawiec et al., 2004; Latawiec et al., 2005b). It is interesting to note that transmission zeros make a subset of control zeros.

### 4. New approach to MVC design

In an attempt to essentially reduce the computational burden of the extreme points and extreme directions method we introduce yet another effective (and much simpler) approach to the MVC design, in which we make use of the equality (Hunek, 2007; Hunek & Latawiec, 2006)

\[
B(q^{-1})B^R(q^{-1}) = I \tag{8}
\]

Consider an LTI \( n_u \)-input \( n_y \)-output system described by the ARMAX model (2). Put \( w = q^{-1} \) and factorize \( B(w) \) to the Smith form \( B(w) = U(w)S(w)V(w) \), where \( U(w) \) and \( V(w) \) are unimodular. Now, \( B^R(w) = V^{-1}(w)S^R(w)U^{-1}(w) \), with determinants of \( U(w) \) and \( V(w) \) being independent of \( w \), that is possible instability of an inverse polynomial matrix \( B^R(w) \) being related to \( S^R(w) \) only. Amazingly, applying the minimum-norm right \( T \)-inverse \( S^R_0(w) = S(w)^T[S(w)S(w)^T]^{-1} \) guarantees that no control zeros except transmission zeros appear in the inverse \( S^R(w) \). (Employing any other inverses, e.g. \( \tau \)- or \( \sigma \)-inverses, causes the control zeros to appear in \( S^R(w) \) in addition to transmission zeros.) This intriguing result has been confirmed in a number of simulations but no formal proof exists, so far. The result confirms the value of the Smith factorization on the one hand, and the \( T \)-inverse on the other.

### 5. Simulation example

Consider the three-input and two-output unstable system described by noise-free deterministic part of model (2) with

\[
B(q^{-1}) = \begin{bmatrix} 2 + 3.8q^{-1} & 2q^{-1} \\ 4q^{-1} & 1 + 1.9q^{-1} \end{bmatrix}, \quad A(q^{-1}) = 500 + 570q^{-1} + 19q^{-2} \quad \text{and} \quad d = 2.
\]

The control zeros type 1, obtained on a basis of \( T \)-inverse of \( B(q^{-1}) \), determine unstable MVC or perfect control of the system. Besides, it is very difficult to find stable MVC/perfect control on the basis of \( \tau \)- and \( \sigma \)-inverses with control zeros type 2 associated. Since the system has one (stable) transmission zero at \( z = 0.1 \), it is impossible to employ the extreme points and extreme directions method. Therefore, we apply our new method of Section 4. Now, after substitution \( w = q^{-1} \) and after Smith
factorization we obtain \( U(w) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \), \( S(w) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w-10 & 0 \end{bmatrix} \), \( V(w) = \begin{bmatrix} 3.8w^2 + 2 & 2w & 1 \\ 0.2 & -0.1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \)

and finally

\[
u(t) = \begin{bmatrix} 0 & 5 & -0.5 \\ 0 & 0 & -1 \\ 1 & -10 - 19q^{-1} & 1 + 3.9q^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & q^{-1} - 10 \end{bmatrix}^{-1} ) \left( y_{ref} (t+dt) - \widehat{H}(q^{-1}) y(t) \right) \]

with specific forms of \( \widehat{F}^{-1} (q^{-1}) \) and \( \widehat{H}^{-1} (q^{-1}) \) not presented here due to their mathematical complexity. Now, for \( y_{1\text{ref}} = 1, y_{2\text{ref}} = 1.5 \) the outputs remain at the setpoint for \( t \geq d = 2 \) under the stabilizing perfect control, whose plots \( u_1(t), u_2(t) \) and \( u_3(t) \) according to equation (9) are shown in Fig. 1. For clarity, we have chosen to show the performance of (noise-free) perfect control rather than MVC.

![Fig. 1. Perfect control plots for the specific example](image)

**Remark.** However, the Smith factorization approach undeniably contributes to the robust MVC design, in the majority cases the application of the control zeros can give much better results (Hunek, under review), giving rise to the extension of the Davison’s theory of minimum phase systems (Davison, 1983). Unfortunately, there exists no formal proof of the above statement and it has been left for future research.

### 6. Conclusions

The Smith factorization approach to the robust minimum variance control has been presented in this paper. The new method appears much better than others, designed by
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authors, namely those based on the extreme points and extreme directions method and second one called minimum-energy. Firstly, it is computationally much simpler and secondly, it works also for the case when transmission zeros are (nongenerically) present in the nonsquare system. Strange enough, the presented method should operate on the $T$-inverse exclusively and any other inverse applied gives rise to the appearance of control zeros. What is also strange, applying the $T$-inverse directly to the polynomial $B(\cdot)$ (rather than to its Smith-factorized form) inevitably ends up with control zeros. Finally, the new approach confirms the Davison’s theory and indicates the need of the introduction of the complementary control zeros theory.

7. References


The book New Approaches in Automation and Robotics offers in 22 chapters a collection of recent developments in automation, robotics as well as control theory. It is dedicated to researchers in science and industry, students, and practicing engineers, who wish to update and enhance their knowledge on modern methods and innovative applications. The authors and editor of this book wish to motivate people, especially under-graduate students, to get involved with the interesting field of robotics and mechatronics. We hope that the ideas and concepts presented in this book are useful for your own work and could contribute to problem solving in similar applications as well. It is clear, however, that the wide area of automation and robotics can only be highlighted at several spots but not completely covered by a single book.

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