5

Stability Analysis of Polynomials with Polynomic Uncertainty

Petr Hušek
Dept. of Control Engineering, Faculty of Electrical Engineering, Czech Technical University in Prague
Czech Republic

1. Introduction

When dealing with systems with parameter uncertainty most attention is paid to robustness analysis of linear time-invariant systems. In literature the most often investigated topic of analysis of linear time-invariant systems with parametric uncertainty is the problem of stability analysis of polynomials whose coefficients depend on uncertain parameters. The aim is to verify that all roots of such a polynomial are located in some prescribed set in complex plane or to find a bound within that uncertain parameters can vary from nominal ones preserving stability. The former problem is studied in this contribution.

The formulations of basic robustness problems and their first solutions for special cases are very old. For example, in the work (Neimark, 1949) some effective techniques for small number of parameters are presented. A powerful result concerning the stability analysis of polynomials with multilinear dependency of its coefficients is given in the book (Zadeh & Desoer, 1963). Also in Siljak's book (Siljak, 1969) special classes of robust stability analysis problems with parametric uncertainty are studied. Nevertheless, the starting point of an intensive interest in this area was the celebrated Kharitonov theorem (Kharitonov, 1978) dealing with interval polynomials. This elegant theorem with surprisingly simple result is considered as the biggest achievement in control theory in last century. When analysing stability of a polynomial with some dependency of its coefficients on interval parameters the solution becomes more complicated. The Edge theorem (Bartlett et al., 1988) claims that for linear (affine) dependency it is sufficient to check polynomials on exposed edges, the Mapping theorem (Zadeh & Desoer, 1963) provides a simplified sufficient stability condition for systems with multilinear parameter dependency.

To date there are only few results solving the problem of robust stability of polynomials with polynomic structure of coefficients (polynomic interval polynomials) that occur very often e.g. as characteristic polynomials in feedback control of uncertain plant with a fixed controller. None of the results is as elegant as those mentioned earlier. There are two basic approaches – algebraic and geometric. The first one is based on utilization of criteria commonly used for stability analysis of fixed polynomials – Hurwitz or Routh criterion – and their generalization for uncertain polynomials. The second one transforms the multidimensional problem in twodimensional test of frequency plot of the polynomial in

complex plane using zero exclusion principle. Very interesting algorithm using the latter approach is based on Bernstein expansion of a multivariate polynomial (Garloff, 1993).
In this chapter an algorithm for stability analysis of polynomials with polynomic parameter dependency based on geometric approach is presented. It consists in determination of a convex polygon overbounding the value set for each frequency and simple performance of the zero exclusion test. The method provides a sufficient stability condition for a continuous-time polynomial with polynomic coefficient dependency. An arbitrary stability region can be chosen.
The presented procedure is demonstrated and compared with the known results on benchmark example - control of Fiat Dedra engine corresponding to 7-th order polynomial with 7 uncertain parameters.

2. State of the art

There is no elegant result on robust stability of polynomic interval polynomial in comparison with interval, affine linear interval or multilinear interval polynomials. There are only few methods, which solve the problem, however almost all of them treat a little different problem and/or are applicable for polynomials dependent only on small number of parameters or polynomials of lower degree.
(De Gaston and Safonov, 1988) determine the stability margin of a multivariate feedback system with uncertainties entering independently into each feedback loop (which corresponds to multilinear parameter uncertainty) using the Mapping theorem. The box of uncertainties is iteratively splitted so that the value of stability margin is improved. The extension to the case of repeated parameters (polynomic parameter uncertainty) is due to (Sideris and de Gaston, 1986). A computational improvement of this method was done by (Sideris and Sanchez Pena, 1989). The algorithm is based on positivity testing of elements appearing in the first column of Routh table. This leads to determination of roots of multivariate polynomial which causes big numerical problems if the number of uncertain parameters and/or degree of the polynomial is even moderate. An improvement of the algorithm using frequency domain splitting is presented in (Chen & Zhou, 2003).
(Vicino et. al., 1990) suggested an algorithm for computing the stability margin in the l_∞ norm, i.e. the radius of the maximal ball in parameter space centered at a stable nominal point preserving stability, for uncertain systems affected by polynomially correlated perturbations. The original constrained nonlinear programming problem, which is generally nonconvex and may admit local extremes, is transformed into a signomial programming problem. An iterative procedure determining a sequence of lower and upper bounds converging to the global extreme is applied.
(Walter and Jaulin, 1994) characterize the set of all the values of the parameters of a linear time-invariant model that are associated with a stable behaviour. A formal Routh table is used to formulate the problem as one of set inversion, which is solved approximately but globally with tools borrowed from interval analysis.
(Kaesbauer, 1993) computes the stability radius for polynomic interval polynomial by solving a system of algebraic equations numerically using the Groebner basis. The method can be practically used up to five or six parameter case.
The most effective algorithm treating the problem of checking stability of polynomials with polynomic parameter uncertainty seems to be the one based on Bernstein expansion (Garloff, 1993) and its improvements (Garloff et al., 1997; Zettler & Garloff, 1998). The

procedure uses suitable properties of the Bernstein form of a multivariate polynomial and test stability by successive subdivision of the original parameter domain and checking positivity of a multivariate polynomial. It can be used in both algebraic (checking positivity of Hurwitz determinant) or geometric (testing the value set) approaches.
Conceptually the same approach is adopted by (Siljak and Stipanovic, 1999). They check robust stability by positivity test of the magnitude of frequency plot by searching minorizing polynomials and using Bernstein expansion. Methods of interval arithmetic are employed in (Malan et al., 1997). Solution of the problem using soft computing methods is presented in (Murdoch et al., 1991).

3. Backgrounds

At first let us introduce the basic terms and general results used in robust stability analysis of linear systems with parametric uncertainty.
DEFINITION 1 (Fixed polynomial) A polynomial $p(s)$ is said to be fixed polynomial of degree n, if

$$p(s)=\sum_{j=0}^{n} a_j s^j = a_n s^n + \cdots + a_1 s + a_0 \,. \tag{1}$$

DEFINITION 2 (Uncertain parameter) An l-dimensional column vector $\mathbf{q}=[q_1,\ldots,q_l]^T \in Q$ represents uncertain parameter. Q is called the uncertainty bounding set. In the whole work

$$Q=\left\{\mathbf{q}\in \Re^l : q_i^- \le q_i \le q_i^+ \text{ for } i=1,2,\ldots,l\right\}, \tag{2}$$

where $q_i^-, q_i^+,\ i=1,2,\ldots,l$ are the specified bounds for the i-th component q_i of $\mathbf{q}$. Such a Q is called a box.
DEFINITION 3 (Uncertain polynomial) A polynomial

$$p(s,\mathbf{q})=\sum_{j=0}^{n} a_j(\mathbf{q}) s^j = a_n(\mathbf{q}) s^n + \cdots + a_1(\mathbf{q}) s + a_0(\mathbf{q});\ \mathbf{q}\in Q \,. \tag{3}$$

is called an uncertain polynomial.
DEFINITION 4 (Polynomic uncertainty structure) An uncertain polynomial (3) is said to have a polynomic uncertainty structure if each coefficient function $a_j(\mathbf{q})$, $j=0,\ldots,n$ is a multivariate polynomial in the components of $\mathbf{q}$.
DEFINITION 5 (Stability, Hurwitz stability) A fixed polynomial $p(s)$ is said to be stable if all its roots lie in the strict left half plane.
DEFINITION 6 (Robust stability) A given family of polynomials $P=\{p(\cdot,\mathbf{q}):\mathbf{q}\in Q\}$ is said to be robustly stable if, for all $\mathbf{q}\in Q$, $p(s,\mathbf{q})$ is stable; that is, for all $\mathbf{q}\in Q$, all roots of $p(s,\mathbf{q})$ lie in the strict left half plane.
THEOREM 1 (Zero exclusion principle)
The family of polynomials P mentioned above of invariant degree is robustly stable if and only if

a. there exists a stable polynomial $p(s,\mathbf{q})\in P$
b. $0\notin p(j\omega,\mathbf{q})$ for all $\omega \ge 0$ ♣

The set $p(j\omega,\mathbf{q})$ for any $\omega > 0$ is called the value set.

The Zero exclusion principle can be used to derive computational procedures for robust stability problems of interval polynomials and polynomials with affine linear, multilinear and polynomic uncertainty. Moreover, for more complicated uncertainty structures where no theoretical results are available the graphical test of zero exclusion can be applied. One can take many points of uncertainty set *Q*, plot the corresponding value sets and visually test if zero is excluded from all of them. The main problem consists in the choice of "sampling" density in some direction of an *l*-dimensional uncertain parameter **q** especially for high values of *l*.

4. Polynomials with quadratic parametric uncertainty

An efficient method analyzing robust stability of polynomials with uncertain coefficients being quadratic functions of interval parameters is presented in this section. A sufficient condition is derived by overbounding the (generally nonconvex) value set by a convex hull (polygon) for an arbitrary point in the complex plane lying on the boundary of chosen stability region and by determination whether zero is excluded from or included in this polygon. This test can be done either in computational or in graphical way. Profiting from appropriate properties of presented procedure the former is recommended especially for high number of parameters. This method can be used in principle for polynomials where the coefficients are arbitrary polynomic functions, which is shown in section 5.

4.1 Basic concept

Let us consider a polynomic interval family of polynomials

$$P(s,\mathbf{q}) = c_n(\mathbf{q})s^n + \cdots + c_1(\mathbf{q})s + c_0(\mathbf{q}),\ \mathbf{q} \in Q \subset \Re^l,\ \mathbf{q} = [q_1,\ldots,q_l]^T$$

$$Q = [q_1^-,q_1^+] \times \cdots \times [q_l^-,q_l^+],\ q_i \in [q_i^-,q_i^+],\ q_i^- < q_i^+,\ i = 1,\ldots,l\,. \tag{4}$$

Let us suppose that each coefficient $c_k(\mathbf{q})$, $k = 0,\ldots,n$ can be expressed as

$$c_k(\mathbf{q}) = \mathbf{q}^T\mathbf{B}^{(k)}\mathbf{q} + \left(\mathbf{d}^{(k)}\right)^T\mathbf{q} + v^{(k)},\ \mathbf{B}^{(k)} \in \Re^{l,l},\ \mathbf{d}^{(k)} \in \Re^l,\ v^{(k)} \in \Re,\ k = 0,\ldots,n\,. \tag{5}$$

Such a function is called a quadratic function and the polynomial $P(s,\mathbf{q})$ is referred to as a quadratic interval polynomial. To avoid dropping in degree, $c_n(\mathbf{q}) \neq 0$ for all $\mathbf{q} \in Q$ is assumed.

In the section if $\mathbf{B} \in \Re^{l,l}$ is a $(l \times l)$ matrix then b_{ij} denotes the element of **B** lying on the position (i, j), if $\mathbf{d} \in \Re^l$ is a vector then d_i denotes the element of **d** lying on the i-th position.

4.2 Determination of a convex polygon

Presented method deals with the value set of $P(s,\mathbf{q})$ evaluated at some complex point $s = s_0 = |s_0|e^{j\psi_0}$. The image $P(s_0,\mathbf{q})$ can be expressed as

$$P(s_0,\mathbf{q}) = \sum_{k=0}^{n} c_k(\mathbf{q})s_0^k = c_{\mathrm{Re}}^{s_0}(\mathbf{q}) + j.c_{\mathrm{Im}}^{s_0}(\mathbf{q}) \tag{6}$$

where $c_{\mathrm{Re}}^{s_0}(\mathbf{q}), c_{\mathrm{Im}}^{s_0}(\mathbf{q})$ are real quadratic functions and are given by

$$c_{\mathrm{Re}}^{s_0}(\mathbf{q}) = \sum_{k=0}^{n} c_k(\mathbf{q}) |s_0|^k \cos(k\psi_0), \quad c_{\mathrm{Im}}^{s_0}(\mathbf{q}) = \sum_{k=0}^{n} c_k(\mathbf{q}) |s_0|^k \sin(k\psi_0) \cdot \tag{7}$$

The idea consists in determining the minimum and maximum differences $h_{\min}^{s_0}(\varphi), h_{\max}^{s_0}(\varphi)$ of the point [0, j0] from the set $P(s_0,\mathbf{q})$ in the complex plane in some direction φ, $\varphi \in [0,\pi]$, respectively (see Fig. 1).

REMARK 1 It is worth noting that the difference is measured from the point [0, j0] in the direction φ, $\varphi \in [0,\pi]$. It means that the difference can be negative (in such a case the difference is measured from the point [0, j0] in the direction $\pi+\varphi$).

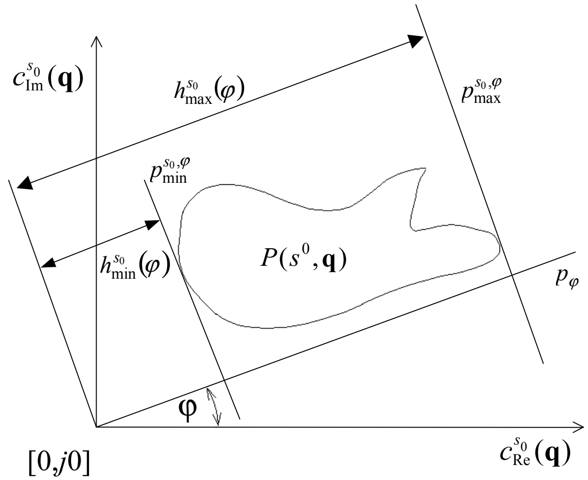


Figure 1. Minimum and maximum distance of $P(s_0,\mathbf{q})$ from [0, j0] in a direction φ

It can be easily shown that finding the minimum and maximum differences is equivalent to finding the minimum and maximum value of the function $c_{\varphi}^{s_0}(\mathbf{q})$,

$$c_{\phi}^{s_0}(\mathbf{q}) = c_{\mathrm{Re}}^{s_0}(\mathbf{q})\cos(\phi) + c_{\mathrm{Im}}^{s_0}(\mathbf{q})\sin(\phi) = \left[c_{\mathrm{Re}}^{s_0}(\mathbf{q}), c_{\mathrm{Im}}^{s_0}(\mathbf{q})\right] . \left[\cos(\phi), \sin(\phi)\right]^T \tag{8}$$

over the set Q.

From (8) it follows that $c_{\varphi}^{s_0}(\mathbf{q})$ is a real quadratic function of q. It means that $c_{\varphi}^{s_0}(\mathbf{q})$ is bounded and $h_{\min}^{s_0}(\varphi), h_{\max}^{s_0}(\varphi)$ are both finite.

The problem of finding extreme values of $c_{\varphi}^{s_0}(\mathbf{q})$ on a box Q is a task of mathematical programming. General formulation of a task of mathematical programming is as follows.

Let us consider the problem of minimization of a function $f_0(\mathbf{x})$, where the constraints are given in the form of inequalities

$$\min\left\{ f_0(\mathbf{x}) \middle| f_j(\mathbf{x}) \le b_j, j = 1,\ldots,m \right\} \tag{9}$$

DEFINITION 7 Let a point ${}^{0}\mathbf{x}$ satisfy all constraints of (9). Let $J({}^{0}\mathbf{x})$ be the set of indices, for which the corresponding constraints are active (i.e., inequality changes to equality):

$$J\left({}^{0}\mathbf{x}\right)=\left\{j \middle| f_j\left({}^{0}\mathbf{x}\right)=b_j\right\} \tag{10}$$

The point ${}^{0}\mathbf{x}$ is said to be a regular point of the set X given by constraints in (9) if the gradients $\nabla f_j({}^{0}\mathbf{x})$ are linearly independent for all $j\in J({}^{0}\mathbf{x})$.

Necessary conditions for the extreme values can be formulated by the following theorem.

THEOREM 2 (Kuhn-Tucker conditions (Kuhn & Tucker, 1951))

Let ${}^{*}\mathbf{x}$ be a regular point of a set X and a function $f_0(\mathbf{x})$ has in some neighbourhood of ${}^{*}\mathbf{x}$ continuous first partial derivatives. If the function $f_0(\mathbf{x})$ has in the point ${}^{*}\mathbf{x}$ the local minimum on X, then there exists a (Lagrange) vector ${}^{*}\boldsymbol{\lambda}\in\Re^m$ such that

$$\begin{gathered}\nabla f_0\left({}^{*}\mathbf{x}\right)+\sum_{r=1}^{m}{}^{*}\lambda_r\nabla f_r\left({}^{*}\mathbf{x}\right)=0\\ {}^{*}\lambda_j\left(f_j\left({}^{*}\mathbf{x}\right)-b_j\right)=0\\ \lambda_j^{*}\geq 0\end{gathered} \tag{11}$$

hold for all $j = 1,...,m$.

REMARK 2 For maximization of a function $f_0(\mathbf{x})$ the last inequality of (11) is replaced by ${}^{*}\lambda_j\leq 0$.

To apply Theorem 2 for solving the problem it is necessary to check whether the preconditions of this theorem are satisfied. As $c_{\varphi}^{s_0}(\mathbf{q})$ is a quadratic function, its first partial derivatives are continuous $\forall\mathbf{q}\in Q$ and the second assumption is satisfied. In our case

$$\begin{aligned}&f_0(\mathbf{q})=c_{\varphi}^{s_0}(\mathbf{q})\\ &f_j(\mathbf{q})=(-1)^{j+1}q_i,\quad i=1,\ldots,l,\ \ j=2i-1,2i\\ &b_j=-q_i^{-}\ \text{ for } j \text{ even}\\ &b_j=\ q_i^{+}\quad\text{for } j \text{ odd}\end{aligned} \tag{12}$$

Then

$$\begin{aligned}&\nabla f_j(\mathbf{q})=(-1)^{j+1}\mathbf{e}^{(i)},\ \ \mathbf{q}\in Q,\ \ j=1,\ldots,2l,\\ &i=\frac{j+1}{2}\ \text{ for } j \text{ odd},\ \ i=\frac{j}{2}\ \text{ for } j\ \text{ even}\end{aligned} \tag{13}$$

where $\mathbf{e}^{(i)} = [0,...,0,1,0,...,0]^T$ with 1 being on the i-th position. Because for any $\mathbf{q}\in Q$ only even or only odd constraints (or none of them) can be active $(q_i^{-}<q_i^{+})\ \forall i=1,\ldots,l$, the gradients $\nabla f_j(\mathbf{q})$ are linearly independent $\forall\mathbf{q}\in Q$, $j\in J(\mathbf{q})$. It means that all points $\mathbf{q}\in Q$ are regular.

Due to Theorem 2 it is necessary to determine the gradient $\nabla c_{\varphi}^{s_0}(\mathbf{q})$. From (8)

$$\nabla c_{\varphi}^{s_0}(\mathbf{q})=\left[\nabla c_{\mathrm{Re}}^{s_0}(\mathbf{q})\ ,\nabla c_{\mathrm{Im}}^{s_0}(\mathbf{q})\right]\ .\left[\cos(\phi),\sin(\phi)\right]^T \tag{14}$$

The components of $\nabla c_k(\mathbf{q})$,

$$\nabla c_k(\mathbf{q}) = \left[\frac{\partial c_k(\mathbf{q})}{\partial q_1}, \ldots, \frac{\partial c_k(\mathbf{q})}{\partial q_l}\right]^T \tag{15}$$

follow from (5):

$$\frac{\partial c_k(\mathbf{q})}{\partial q_i} = 2b_{ii}^{(k)} q_i + \sum_{\substack{r=1 \\ r \neq i}}^{l} \left(b_{ir}^{(k)} + b_{ri}^{(k)}\right) q_r, \; k = 0, \ldots, n \quad i = 1, \ldots, l \tag{16}$$

From (7)

$$\begin{aligned} \nabla c_{\mathrm{Re}}^{s_0}(\mathbf{q}) &= \sum_{k=0}^{n} \nabla c_k(\mathbf{q}) |s_0|^k \cos(k\psi_0) \\ \nabla c_{\mathrm{Im}}^{s_0}(\mathbf{q}) &= \sum_{k=0}^{n} \nabla c_k(\mathbf{q}) |s_0|^k \sin(k\psi_0) \end{aligned} \tag{17}$$

After substituting (12), (13), (14), (15), (16) and (17) to (11) the following system of equations and inequalities is obtained:

$$\left[\begin{array}{ccc|ccccc} W_{11} & \cdots & W_{1l} & 1 & -1 & 0 & \cdots & \cdots \\ \vdots & \ddots & \vdots & 0 & 0 & 1 & -1 & \cdots \\ & \ddots & & \vdots & \ddots & & \ddots & \ddots \\ W_{l1} & \cdots & W_{ll} & 0 & \cdots & 0 & 1 & -1 \end{array}\right] \cdot \begin{bmatrix} q_1 \\ \vdots \\ q_l \\ \lambda_1 \\ \vdots \\ \lambda_{2l} \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_l \end{bmatrix} \tag{18}$$

$$\begin{aligned} \lambda_1\left(q_1 - q_1^+\right) &= 0 \\ \lambda_2\left(-q_1 - q_1^-\right) &= 0 \\ \lambda_3\left(q_2 - q_2^+\right) &= 0 \\ \lambda_4\left(-q_2 - q_2^-\right) &= 0 \\ &\vdots \\ \lambda_{2l-1}\left(q_l - q_l^+\right) &= 0 \\ \lambda_{2l}\left(-q_l - q_l^-\right) &= 0 \end{aligned} \tag{19}$$

$$\begin{aligned} \lambda_1, \ldots, \lambda_{2l} \geq 0 \;\; \text{for minimization} \\ \lambda_1, \ldots, \lambda_{2l} \leq 0 \;\; \text{for maximization} \end{aligned} \tag{6.1}$$

where

$$W_{uv}=\left[\sum_{k=0}^{n}\left(b_{uv}^{(k)}+b_{vu}^{(k)}\right).\left|s_0\right|^k\cos\left(k\psi_0\right)\right].\cos(\varphi)+\left[\sum_{k=0}^{n}\left(b_{uv}^{(k)}+b_{vu}^{(k)}\right).\left|s_0\right|^k\sin\left(k\psi_0\right)\right].\sin(\varphi)$$

$$w_u=\left[\sum_{k=0}^{n}d_u^{(k)}.\left|s_0\right|^k\cos\left(k\psi_0\right)\right].\cos(\varphi)+\left[\sum_{k=0}^{n}d_u^{(k)}.\left|s_0\right|^k\sin\left(k\psi_0\right)\right].\sin(\varphi)$$

$$u,v=1,\ldots,l.$$

The important fact is that the equation (18) is linear. The computational way of solving the system (18-19) runs as follows. First all the solutions of (19) are determined. This corresponds to determining of all the parts of the box Q – interior and all the parts of the boundary of Q (all manifolds with the dimension i, i = 0,..., l-1 containing only points on the boundary of Q). Each solution of (19) corresponds to 2l linear equations (from (19) it follows that at least one of λ_{2i-1}, λ_{2i}, i = 1,..., l has to equal zero; if $\lambda_{2i-1} = 0$ then either $\lambda_{2i} = 0$ or $q_i = - q_i^-$, if $\lambda_{2i} = 0$ then either $\lambda_{2i-1} = 0$ or $q_i = q_i^+$ i = 1,..., l). These 2l equations together with l equations of (18) form 3l linearly independent linear equations for 3l unknown variables. It means that there exists a unique solution ($^*\boldsymbol{\lambda}$,$^*\mathbf{q}$) (for each solution of (19)) of system (18-19). Denote by $T_{\min}$ ($T_{\max}$) the set of t for which these conditions are satisfied,

$$\begin{aligned} T_{\min} &= \left\{t{:}\,{}^*\mathbf{q}^{(t)}\in Q,{}^*\lambda_j^{(t)}\geq 0\ \forall j=1,\ldots,2l\right\} \\ T_{\max} &= \left\{t{:}\,{}^*\mathbf{q}^{(t)}\in Q,{}^*\lambda_j^{(t)}\leq 0\ \forall j=1,\ldots,2l\right\} \end{aligned} \tag{20}$$

Then

$$\begin{aligned} h_{\min}^{s_0}(\varphi) &= \min_{t\in T_{\min}}\left[c_\varphi^{s_0}\left({}^*\mathbf{q}^{(t)}\right)\right] \\ h_{\max}^{s_0}(\varphi) &= \max_{t\in T_{\max}}\left[c_\varphi^{s_0}\left({}^*\mathbf{q}^{(t)}\right)\right] \end{aligned} \tag{21}$$

The minimum and maximum differences indicate that the set $P(s_0,\mathbf{q})$ lies in the complex plane in the space between the lines $p_{\min}^{s_0,\varphi}$ and $p_{\max}^{s_0,\varphi}$:

$$\begin{aligned} p_{\min}^{s_0,\varphi}\ &:\ c_{\mathrm{Im}}^{s_0}(\mathbf{q})=-\frac{1}{\tan(\varphi)}c_{\mathrm{Re}}^{s_0}(\mathbf{q})+\frac{h_{\min}^{s_0}(\varphi)}{\sin(\varphi)} \\ p_{\max}^{s_0,\varphi}\ &:\ c_{\mathrm{Im}}^{s_0}(\mathbf{q})=-\frac{1}{\tan(\varphi)}c_{\mathrm{Re}}^{s_0}(\mathbf{q})+\frac{h_{\max}^{s_0}(\varphi)}{\sin(\varphi)} \end{aligned} \tag{22}$$

In order to determine a convex hull overbounding the set $P(s_0,\mathbf{q})$, $\mathbf{q}\in$Q, the procedure described above is performed for a set of $\varphi_r\in\Phi$,

$$\Phi=\left\{\begin{matrix}\varphi_r:0\leq\varphi_1\leq\cdots\leq\varphi_{R-1}\leq\varphi_R\leq\pi, \\ r=1,\ldots,R\end{matrix}\right\} \tag{23}$$

It means that the system (18-19) is solved for a set of φ. The higher the number R is, the "more tight" convex hull is obtained.

If one wants to determine the convex polygon computationally the set $V_\Phi(s_0)$ of the intersections of the following lines has to be determined:

$$
\begin{aligned}
V_{\Phi}(s_0) &= \left\{ S_m^{s_0} : m = 1,\ldots,2R \right\} \\
V_r^{s_0} &= \mathrm{insec}\left(p_{\min}^{s_0,\varphi_r}, p_{\min}^{s_0,\varphi_{r+1}}\right) \\
V_R^{s_0} &= \mathrm{insec}\left(p_{\min}^{s_0,\varphi_R}, p_{\max}^{s_0,\varphi_1}\right) \\
V_{r+R}^{s_0} &= \mathrm{insec}\left(p_{\max}^{s_0,\varphi_r}, p_{\max}^{s_0,\varphi_{r+1}}\right) \\
V_{2R}^{s_0} &= \mathrm{insec}\left(p_{\max}^{s_0,\varphi_R}, p_{\min}^{s_0,\varphi_1}\right) \\
r &= 1,\ldots,R-1
\end{aligned}
\tag{24}
$$

where insec(p_x, p_y) denotes the intersection of the lines p_x and p_y (see Fig. 2).

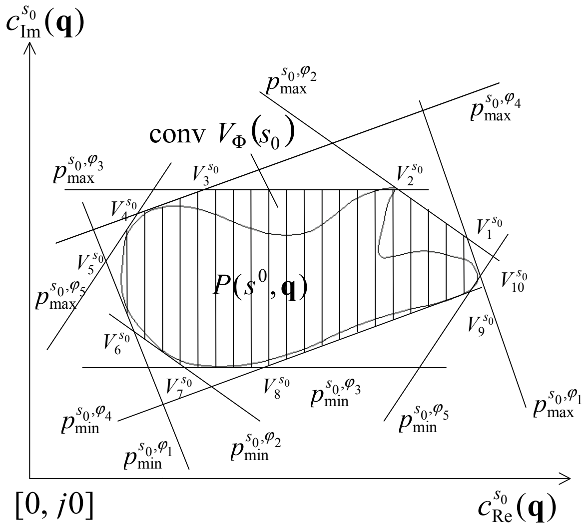


Figure 1. Convex hull $V_{\Phi}(s_0)$ for R = 5

The coordinates of intersections are given by

$$
\mathrm{insec}\left(p_{\mathrm{term}}^{s_0,\varphi_m}, p_{\mathrm{term}}^{s_0,\varphi_{m+1}}\right) = \begin{bmatrix} \dfrac{h_{\mathrm{term}}^{s_0}(\varphi_2)\sin(\varphi_1) - h_{\mathrm{term}}^{s_0}(\varphi_1)\sin(\varphi_2)}{\sin(\varphi_1 - \varphi_2)} \\ \dfrac{h_{\mathrm{term}}^{s_0}(\varphi_2)\cos(\varphi_1) - h_{\mathrm{term}}^{s_0}(\varphi_1)\cos(\varphi_2)}{\sin(\varphi_1 - \varphi_2)} \end{bmatrix}^T
\tag{25}
$$

where term stands for min or max.

Now the key theorems can be stated.

THEOREM 3 (Convex polygons overbounding the value set)

Denote by conv A the convex hull of a set A. Then

$$
P(s_0,\mathbf{q}) \subseteq \mathrm{conv}\ V_{\Phi}(s_0) \forall s_0 \in C
\tag{26}
$$

Using Theorem 1 the Zero exclusion principle gives a necessary condition for stability of a family of polynomials (4).

THEOREM 4 (Sufficient robust stability condition)
The family of polynomials (4) of constant degree containing at least one stable polynomial is robustly stable with respect to S if

$$0 \notin \text{conv } V_\Phi(s_0) \text{ for all } s_0 \in \partial S \tag{27}$$

where ∂S denotes the boundary of S.

The zero exclusion test can be performed in both graphical and computational way. The latter is recommended as described below because of saving a lot of time.
THEOREM 5
$0 \notin \text{conv } V_\Phi(s_0)$ if and only if there exists at least one $\varphi \in \Phi$, such that

$$h_{\min}^{s_0}(\varphi) \geq 0 \wedge h_{\max}^{s_0}(\varphi) \geq 0 \text{ or } h_{\min}^{s_0}(\varphi) \leq 0 \wedge h_{\max}^{s_0}(\varphi) \leq 0 \tag{28}$$

Theorem 5 makes it possible to decide about zero exclusion or inclusion without computing the set of intersections $V_\Phi(s_0)$. Proofs of all three theorems are evident from the construction of convex polygons and Zero exclusion theorem.
Let us illustrate the described procedure of checking robust stability of quadratic interval polynomials on two examples. As arbitrary stability region can be chosen a discrete-time uncertain polynomial will be considered at first.
EXAMPLE 1 Let a family of discrete-time polynomials be given by

$$P(z,\mathbf{q}) = c_2(\mathbf{q})z^2 + c_1(\mathbf{q})z + c_0(\mathbf{q})$$

where

$$\mathbf{q} = [q_1, q_2]^T, \; q_i \in [0,1]$$

and

$$\begin{aligned} c_2(\mathbf{q}) &= 1 \\ c_1(\mathbf{q}) &= 0.2 \cdot q_2 - 0.5 \cdot q_2^2 + 0.1 \cdot q_1 \cdot q_2 \\ c_0(\mathbf{q}) &= -0.3 \cdot q_1 + 0.2 \cdot q_1^2 - 0.5 \cdot q_2^2 + q_1 \cdot q_2 \end{aligned}$$

The question is whether this family of polynomials is Schur stable.
In this case the stability region S is the unit circle, therefore its boundary $\partial S = e^{j\omega}$, $\omega \in [0,2\pi]$. The Zero exclusion principle will be tested graphically. Due to symmetry it is sufficient to plot the value set only for the points $s_0 = e^{j\omega}$, $\omega \in [0,\pi]$. The corresponding plot of the value sets and their convex hulls is shown in Fig. 3 and Fig. 4 (R = 6) respectively. As $0 \notin V_\Phi(s_0)$ for all $s_0 \in \partial S$, the polynomial $P(z,\mathbf{q})$ is robustly Schur stable. In Fig. 5 and Fig. 6 the value set and the convex hull for $s_0 = e^{j\pi/3}$ and different number of angles φ_r is plotted (R = 4 and R = 14 respectively).

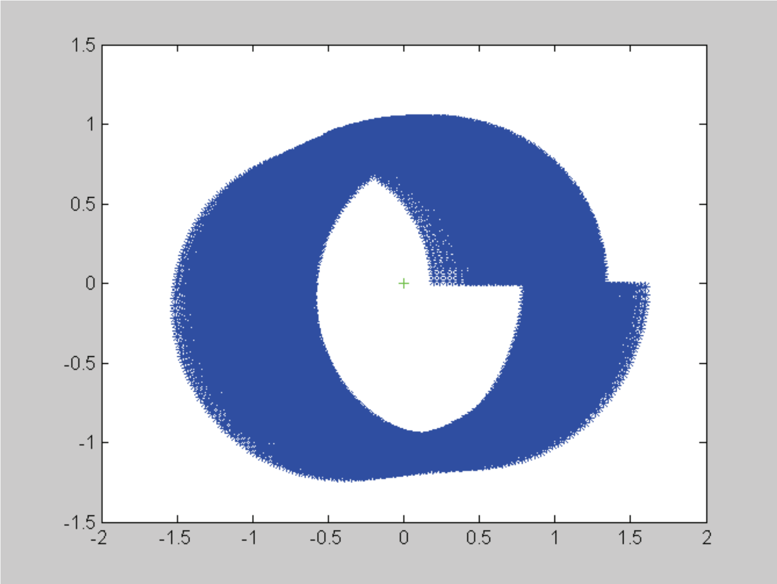


Figure 2. Plot of the value set for $s_0 = e^{j\omega}, \omega \in [0, \pi]$

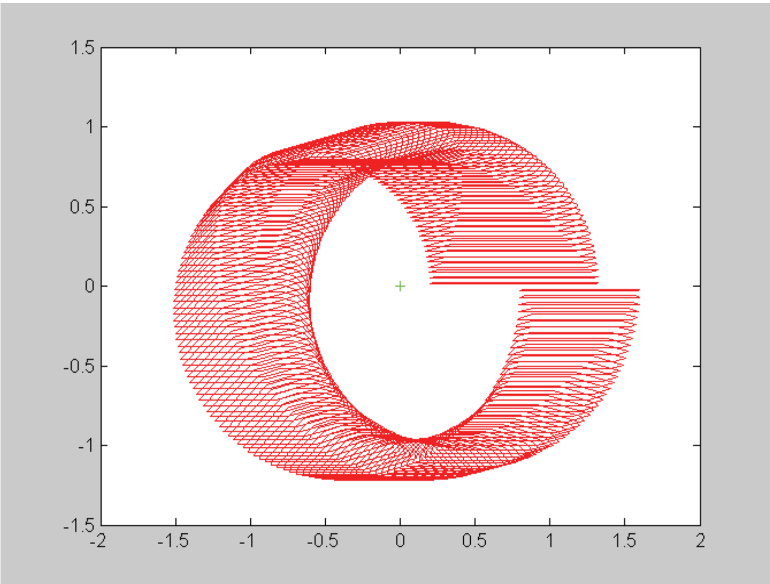


Figure 3. Plot of the convex hulls of the value set

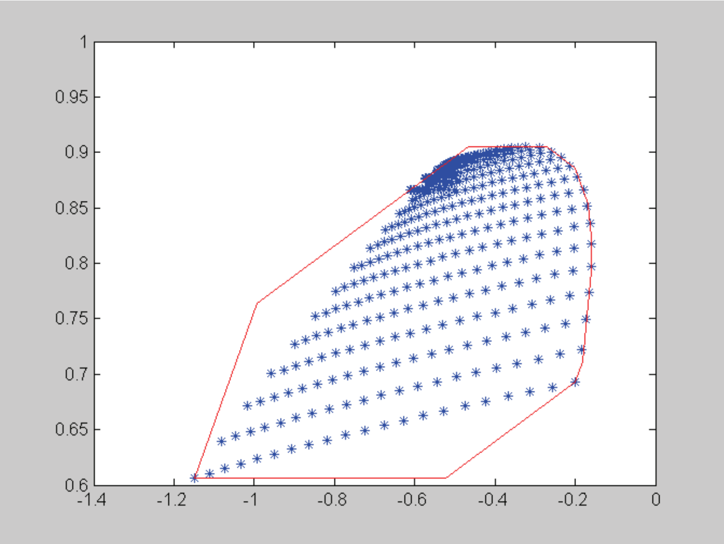


Figure 4. The value set and the convex hull for $s_0 = e^{j\pi/3}$ and $R = 4$

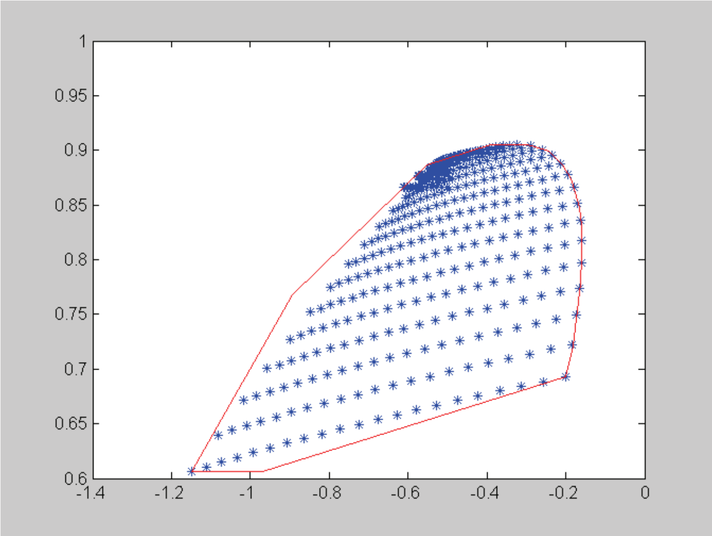


Figure 5. The value set and the convex hull for $s_0 = e^{j\pi/3}$ and $R = 14$

EXAMPLE 2 Let a family of continuous-time polynomials be given by

$$P(s,\mathbf{q}) = c_3(\mathbf{q})s^3 + c_2(\mathbf{q})s^2 + c_1(\mathbf{q})s + c_0(\mathbf{q})$$

where

$$\mathbf{q} = [q_1, q_2]^T, \; q_i \in [0,1]$$

and

$$c_3(\mathbf{q}) = 1$$
$$c_2(\mathbf{q}) = 7.7640 + 6.6486q_1 + 7.0064q_2 + 9.9945q_1^2 + 7.0357q_2^2 + 5.6677q_1 \cdot q_2$$
$$c_1(\mathbf{q}) = 4.8935 + 3.6537q_1 + 9.8271q_2 + 9.6164q_1^2 + 4.8496q_2^2 + 8.2301q_1 \cdot q_2$$
$$c_0(\mathbf{q}) = 1.8590 + 1.4004q_1 + 8.0664q_2 + 0.5886q_1^2 + 1.1461q_2^2 + 6.7395q_1 \cdot q_2$$

The question is whether this family of polynomials is Hurwitz stable.

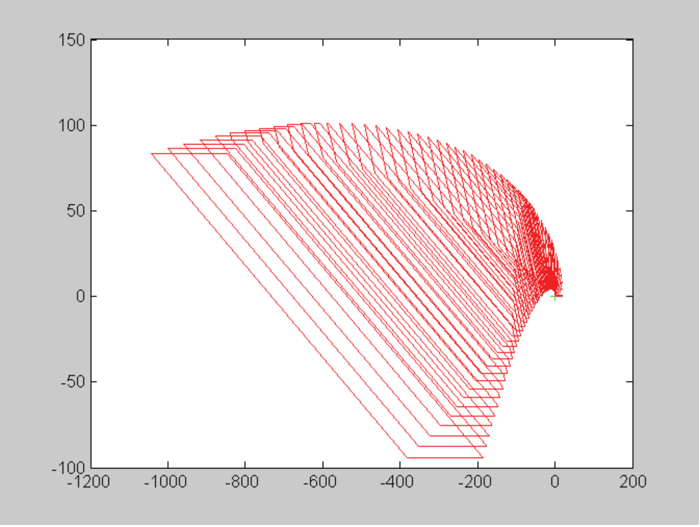


Figure 7. Plot of the convex hulls of the value sets for $s_0 = j\omega$, $\omega \in [0,5]$

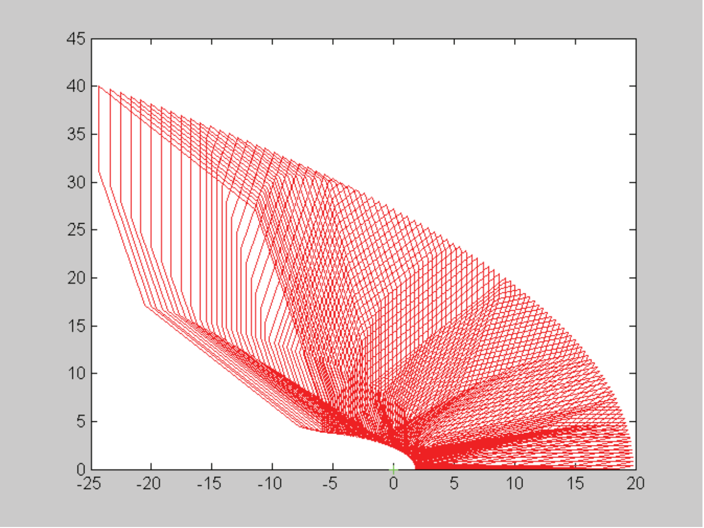


Figure 8. Plot of the convex hulls of the value sets for $s_0 = j\omega$, $\omega \in [0,1]$

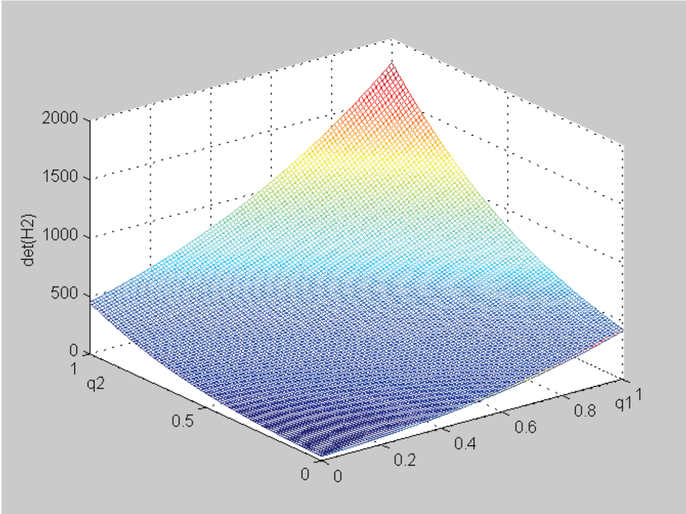


Figure 9. Plot of the determinant of the matrix $\mathbf{H}_2(\mathbf{q})$

Here the stability region S is the imaginary axis, therefore the boundary $\partial S = j\omega$, $\omega \in [-\infty, \infty]$. Due to symmetry it is sufficient to plot the value set only for $s_0 = j\omega$, $\omega \in [0, \infty]$. The corresponding plot of the convex hulls for $\omega \in [0,5]$ is shown in Fig. 7. As from this figure it is not apparent, whether zero is included or not, the same plot for $\omega \in [0,1]$ is shown in Fig. 8. From that it is clear that $0 \notin V_\Phi(s_0)$ for all $s_0 \in \partial S$. The polynomial $P(s,\mathbf{q})$ is robustly Hurwitz stable.

The obtained result can be confirmed by plotting the determinant of the (n-1)-th order Hurwitz matrix $\mathbf{H}_2(\mathbf{q})$ and checking its positivity as $c_0(\mathbf{q})$ is positive for admissible $\mathbf{q}$ evidently. Fig. 9 confirms the obtained result.

4. Polynomials of general polynomic parameter uncertainty

The result obtained in Theorem 5 is applicable for uncertain polynomials with arbitrary polynomic parameter dependency as well. In such case it is necessary to determine if the function $c_\varphi^{s_0}(\mathbf{q})$ is positive or negative on the set Q or it allows both positive and negative values on this set, i.e., if there exists a $\mathbf{q}^1 \in Q$ such that $c_\varphi^{s_0}(\mathbf{q}^1) > 0$ and $\mathbf{q}^2 \in Q$ such that $c_\varphi^{s_0}(\mathbf{q}^2) < 0$. Since $c_\varphi^{s_0}(\mathbf{q})$ is a polynomic function its positivity can be tested by effective algorithm of Bernstein expansion (Garloff, 1993).

The algorithm gives only sufficient stability condition. If for all $s_0 \in \partial S$ at least one φ_r is determined, such that the function $c_\varphi^{s_0}(\mathbf{q})$ is only positive or only negative on the set Q, then the origin is excluded from the convex hulls of value sets for all $s_0 \in \partial S$ and therefore also from the value set itself and the family of polynomials is stable. If not, it is not possible to decide about robust stability of the family.

The main advantage of this algorithm is that the number of coefficients of multivariate polynomic function $c_{\varphi}^{s_0}(\mathbf{q})$ is considerably smaller than the of Hurwitz determinant $\det(\mathbf{H}_{n-1}(\mathbf{q}))$ especially for higher number of uncertain parameters (however still moderate) because using the value set algorithm only the coefficients of tested polynomial are needed to store. For example, a polynomial of degree n = 5 with l = 4 uncertain parameters with highest degree equal to 4 appearing in each variable in each original coefficient contains generally 120 coefficients. The determinant of (n-1)-th order Hurwitz matrix, which has to be tested for positivity, contains generally 83521 coefficients. If the number of parameters is doubled (l = 8), the uncertain polynomial contains 240 coefficients, but the determinant of (n-1)-th order Hurwitz matrix contains huge $6.98 \cdot 10^9$ coefficients which is out of memory for standard computers. Therefore this algorithm can deal with much larger problems. This is demonstrated on the benchmark example of Fiat Dedra engine.
The proposed algorithm will be demonstrated on some examples and its efficiency compared with the of original application of algorithm of Bernstein expansion.
EXAMPLE 3 Let a family of continuous-time polynomials be given by

$$p(s,\mathbf{q}) = c_3(\mathbf{q})s^3 + c_2(\mathbf{q})s^2 + c_1(\mathbf{q})s + c_0(\mathbf{q})$$

where

$$\mathbf{q} = [q_1, q_2, q_3]^T, \quad q_i \in [0,1], \; i = 1,2,3$$

and

$$\begin{aligned}
c_3(\mathbf{q}) = {} & q_1 + q_1^2 + 3q_2 + 1q_1q_2 + 5q_1^2q_2 + 2q_1q_2^2 + q_1^2q_2^2 + 3q_3 + 4q_1q_3 + q_1^2q_3 + 3q_2q_3 \\
& + 2q_1q_2q_3 + q_1^2q_2q_3 + 4q_2^2q_3 + 4q_1q_2^2q_3 + 4q_1^2q_2^2q_3 + 3q_1q_3^2 + 2q_1^2q_3^2 + 3q_2q_3^2 \\
& + 5q_1q_2q_3^2 + 3q_1^2q_2q_3^2 + 2q_2^2q_3^2 + 4q_1q_2^2q_3^2 + 4q_1^2q_2^2q_3^2; \\
c_2(\mathbf{q}) = {} & 8 + 3q_1 + 3q_1^2 + 3q_2 + 5q_1q_2 + 2q_1^2q_2 + 10q_2^2 + 3q_1q_2^2 + 8q_1^2q_2^2 + 9q_3 + 3q_1q_3 \\
& + q_1^2q_3 + 3q_2q_3 + 7q_1q_2q_3 + 5q_2^2q_3 + 6q_1q_2^2q_3 + 7q_1^2q_2^2q_3 + 6q_3^2 + 7q_1q_3^2 \\
& + 8q_1^2q_3^2 + 9q_2q_3^2 + 10q_1q_2q_3^2 + 9q_1^2q_2q_3^2 + 2q_2^2q_3^2 + 10q_1q_2^2q_3^2 + 9q_1^2q_2^2q_3^2; \\
c_1(\mathbf{q}) = {} & 6 + 7q_1 + q_1^2 + 8q_2 + 5q_1q_2 + 9q_1^2q_2 + q_1q_2^2 + 7q_1^2q_2^2 + 6q_3 + 9q_1q_3 + 5q_1^2q_3 \\
& + 5q_2q_3 + 4q_1q_2q_3 + 4q_1^2q_2q_3 + 4q_2^2q_3 + 9q_1q_2^2q_3 + 8q_1^2q_2^2q_3 + 9q_3^2 + 8q_1q_3^2 \\
& + 9q_1^2q_3^2 + 8q_2q_3^2 + 4q_1q_2q_3^2 + 4q_1^2q_2q_3^2 + 2q_2^2q_3^2 + 4q_1^2q_2^2q_3^2; \\
c_0(\mathbf{q}) = {} & 6 + q_1 + q_1^2 + 6q_2 + 9q_1q_2 + 4q_1^2q_2 + 7q_2^2 + q_1q_2^2 + 5q_3 + 9q_1q_3 + 8q_1^2q_3 \\
& + 8q_2q_3 + 2q_1q_2q_3 + 7q_1^2q_2q_3 + 2q_2^2q_3 + 8q_3 + q_1q_2^2q_3 + 2q_1^2q_2^2q_3 + 2q_3^2 \\
& + 5q_1q_3^2 + q_1^2q_3^2 + 6q_2q_3^2 + 2q_1q_2q_3^2 + 9q_1^2q_2q_3^2 + 3q_2^2q_3^2 + 5q_1q_2^2q_3^2 + 8q_1^2q_2^2q_3^2
\end{aligned}$$

The dependency of polynomial coefficients $c_j(\mathbf{q}), j = 0,\ldots,3$ is no longer quadratic and Bernstein algorithm will be used to check positivity or negativity of all the distances. The algorithm checks in 0.34 seconds that for $\omega \in [0,2]$ with step 0.01 (R=10) the origin is excluded from all the convex hulls of value sets and therefore also from the value set itself and the family of polynomials is stable. This result is also confirmed by plotting the value set (Fig. 10). The Bernstein algorithm (Zettler & Garloff, 1998) applied on value sets gives the same result in 0.94s. The algorithm of Bernstein expansion can be also employed on positivity test of Hurwitz determinant. Using symbolic computations for determination of determinant of Hurwitz matrix the Bernstein algorithm reports the same result after 3.54s.

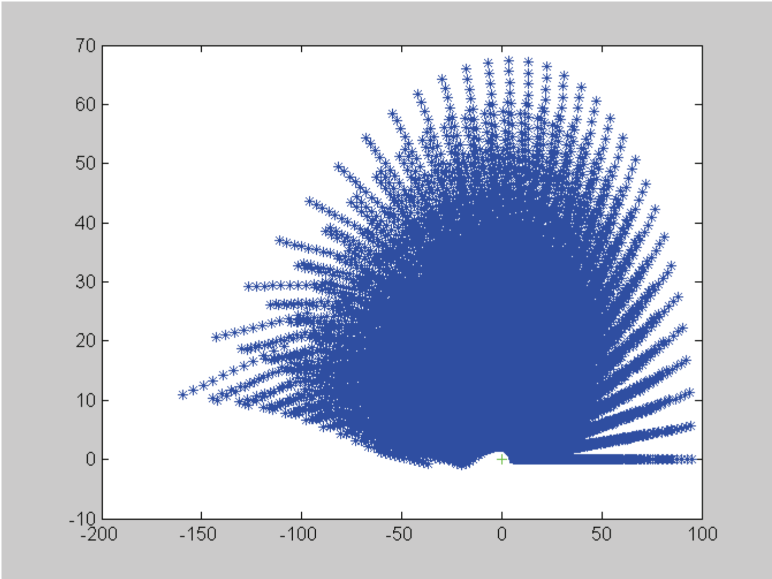


Figure 10. Plot of the value sets of $P(s,\mathbf{q})$ for $\omega \in [0,1.5]$

5. Fiat-Dedra engine

Let us consider a model of the Fiat Dedra engine given in (Barmish, 1994). The focal point is the idle speed control problem, which is particularly important for city driving; that is, fuel economy depends strongly on engine performance when idling.
The model has 7 uncertain parameters and a design of a fixed output controller leads to characteristic polynomial of 7-th order,

$$p(s,\mathbf{q}) = \sum_{j=0}^{7} a_j(\mathbf{q})\, s^j \tag{29}$$

The coefficients $a_j(\mathbf{q})$, j = 0,…,7 being polynomic functions of the parameters q_i, i = 1,...,7 are listed in (Barmish, 1994).
The parameters and the frequency are supposed to vary inside the following intervals:

$$\begin{aligned} &q_1 \in [2.1608,\ 3.4329];\ q_2 \in [0.1027,\ 0.1627];\ q_3 \in [0.0357,\ 0.1139];\\ &q_4 \in [0.2539,\ 0.5607];\ q_5 \in [0.0100,\ 0.0208];\ q_6 \in [2.0247,\ 4.4962];\\ &q_7 \in [1.0000,\ 10.000];\ \omega \in [0.0000,\ 2.3410] \end{aligned} \tag{30}$$

The question is whether the uncertain polynomial (29) is robustly stable for the parameters and frequency given in (30).
Firstly it has to be noted that this problem is relatively large and it is not possible to compute the determinant of the 6-th order Hurwitz matrix because storage capacity of a standard computer is too low to store all its coefficients.
The frequency step was chosen 0.01, the sufficient number of direction angles was 10. The described algorithm reports in 5.53s that the characteristic polynomial (29) is stable that corresponds to the result obtained by the Bernstein expansion (Zettler and Garloff, 1998) in 7.48s. All the computations were performed on a Pentium 4 CPU 3GHz 504MB RAM.

7. Conclusion

The algorithm checking robust stability of polynomials with polynomic dependency of its coefficients on vector interval parameter was presented. The method is based on testing the value set in frequency domain. The value set evaluated in a point lying on the boundary of stability region is overbounded by a convex polygon. The zero exclusion test is performed by positivity checking of multivariate polynomic functions using the Bernstein algorithm. The procedure results in sufficient stability condition. The main advantage of the presented algorithm over those based on computation of Hurwitz determinant consists in its capability of treating relatively large problems because of the low requirements on computer storage capacity. Moreover, arbitrary stability region can be chosen. Efficiency of the algorithm was verified on the benchmark example of the Fiat Dedra engine control by comparison with the Bernstein expansion algorithm.

8. Acknowledgements

This work has been supported by the project INGO 1P2007LA297, Research Program MSM6840770038 (sponsored by the Ministry of Education of the Czech Republic) and the project 1H-PK/22 (sponsored by Ministry of Industry and Trade of the Czech Republic).

9. References

Ackermann, J. (1993). *Robust Control, Systems with Uncertain Physical Parameters*, Springer, ISBN 0-387-19843-1, London

Barmish, B.R. (1994). *New Tools for Robustness of Linear Systems*. Macmillan Publishing Company, ISBN 0-02-306005-7, New York

Bartlett, A.C., Hollot, C.V. & Lin, H. (1988). Root Location of an Entire Polytope of Polynomials: It suffices to check the edges, *Mathematics of Controls, Signals and Systems*, Vol. 1, No. 1, pp. 61-71, ISSN 0932-4194

Bhattacharyya, S. P., Chapellat, H. & Keel, L. H. (1995). *Robust Control: the Parametric Approach*, Prentice-Hall Inc, ISBN 978-0137815760, New York

Garloff, J. (1985). Convergent Bounds for the Range of Multivariate Polynomials. *Interval Mathematics, Lecture Notes in Computer Science*, K. Nickel (Ed.), pp. 37-56, Springer Verlag, ISBN:3-387-16437-5, Berlin

Garloff, J. (1993). The Bernstein algorithm. *Interval Computations*, Vol. 2, No. 6, pp. 154-168, ISSN 0135-4868

Garloff, J., Graf, B. & Zettler, M. (1997). Speeding Up an Algorithm for Checking Robust Stability of Polynomials. *Fachhochschule Konstanz, Internal Report*

Gaston de, R.R. & Safonov, M.G. (1988). Exact Calculation of the Multiloop Stability Margin. *IEEE Transactions on Automatic Control,* Vol. 33, No. 2, pp. 156-171, ISSN 0018-9286

Chen, X. & Zhou, K. (2003). Fast parallel-frequency-sweeping algorithms for robust D-stability, *IEEE Transactions on Circuits and Systems,* Vol. 50, No. 3, pp. 418-428, ISSN 057-7122

Kaesbauer, D. (1993). On Robust Stability of Polynomials with Polynomial Parameter Dependency: Two/three parameter cases, *Automatica,* Vol. 29., No. 1, pp. 215-217, ISSN 0005-1098

Kharitonov, V. L. (1978). Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differencialnyje Uravnenija*, Vol. 14, No. 11, pp. 2086-2088, ISSN 0374 0641

Kuhn, H. W. & Tucker, A. (1951). Nonlinear Programming, *Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley

Malan, S., Milanese M. & Taragna, M. (1997). Robust Analysis and Design of Control Systems Using Interval Arithmetics, *Automatica,* Vol. 33, No. 7, pp. 1363-1372, ISSN 0005-1098

Murdock, T.M., Schmitendorf, W.E. & Forrest, S. (1991). Use of a Genetic Algorithm to Analyze Robust Stability Problems, *Proceedings of American Control Conference*, pp. 886-889, Boston, MA

Neimark,Y. I. (1949). Stability of Linearized Systems, *Aeronautical Engineering Academy*, Leningrad, USSR

Sideris, A. & de Gaston, R.R. (1986). Multivariable Stability Margin Calculation with Uncertain Correlated Parameters, *Proceedings of IEEE Conference on. Decision and Control*, Athens, Greece

Sideris, A. & Sánchez Peňa, R.S. (1989). Fast Computation of the Multivariate Stability Margin for Real Interrelated Uncertain Parameters, *IEEE Transactions on Automatic Control,* Vol. 34., No. 12, pp. 1272-1276, ISSN 0018-9286

Siljak, D. D. (1969). *Nonlinear Systems*, Wiley, New York

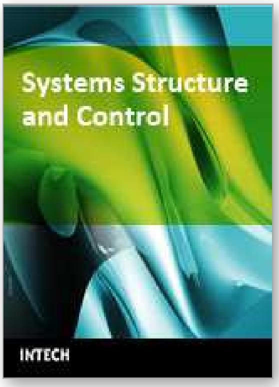
Siljak, D. D. and Stipanovic, D. M. (1999). Robust D-stability via positivity, *Automatica*, Vol. 35, No. 8, pp. 1477-1484, ISSN 0005-1098

Vicino, A., Tesi, A. & Milanese, M. (1990). Computation of Nonconservative Stability Perturbation Bounds for Systems with Nonlinearly Correlated Uncertainties, *IEEE Transactions on Automatic Control,* Vol. 35., No. 7, pp. 835-841, ISSN 0018-9286

Walter, E. and Jaulin, L. (1994). Guaranteed Characterization of Stability Domains Via Set Inversion, *IEEE Transactions on Automatic Control,* Vol. 39, No. 4, pp. 886-889, ISSN 0018-9286

Zadeh, L. A. and Desoer, C. A. (1963). *Linear Systems Theory,* McGraw Hill Book Co., New York

Zettler, M. and Garloff, J. (1998). Robustness Analysis of Polynomials with Polynomial Parameter Dependency Using Bernstein Expansion, *IEEE Transactions on Automatic Control*, Vol. 43, No. 3, pp. 425-431, ISSN 0018-9286



Systems Structure and Control
Edited by Petr Husek

ISBN 978-953-7619-05-3
Hard cover, 248 pages
Publisher InTech
Published online 01, August, 2008
Published in print edition August, 2008

The title of the book System, Structure and Control encompasses broad field of theory and applications of many different control approaches applied on different classes of dynamic systems. Output and state feedback control include among others robust control, optimal control or intelligent control methods such as fuzzy or neural network approach, dynamic systems are e.g. linear or nonlinear with or without time delay, fixed or uncertain, onedimensional or multidimensional. The applications cover all branches of human activities including any kind of industry, economics, biology, social sciences etc.